A Model of Task Encroachment in the Labour Market*

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Abstract

I build a new model to study the impact of technological change on the labour market, based on a simple claim – that new technologies will be able to perform *more* types of tasks in the future. I call this 'task encroachment'. To explore this process, I use a new distinction between two types of capital – 'complementing' capital ('c-capital') and 'substituting' capital ('s-capital'). As the quantity and productivity of s-capital increases, it erodes the set of tasks in which labour is complemented by c-capital. In a static version of the model, this process drives down relative wages and the labour share of income. In a dynamic model, as s-capital is accumulated, labour is driven out the economy and absolute wages decline to zero. I show that c-capital has an important role as a countervailing force against this immiseration of labour.

JEL: J20, J21, J23

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Traditionally the economic literature that explored the consequences of technological change on the labour market tended to support an optimistic view about the threat of automation, though the reasons for that optimism changed over time. In the early literature, the dominant framework was the "canonical" or "textbook"

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model (Acemoglu and Autor 2011; Atkinson 2008). Here, the labour of two distinct skill groups was typically combined in a constant elasticity of substitution production function to produce output. In this model, by construction, it was assumed that technology complements *all* types of worker. In this framework, it is not possible for any workers to be made worse-off by technological change.[1]

In the more recent 'task-based' literature, the reason for any optimism was for different. In these models, technology no longer directly complements all types of workers, but instead only complements *particular* types of workers who are able to perform tasks that cannot be automated. In turn, technology substitutes for those workers who perform tasks that can be automated. Now it is possible that certain workers are made worse-off by technological change (see, for instance, Autor and Acemoglu 2011). In this framework, any optimism instead relies upon the more nuanced claim that there exists a large set of types of tasks that cannot be automated. In short, while the canonical model supported an optimistic view about the threat of automation by making the strong assumption that technology cannot substitute for labour, early task-based models tended to support an optimistic view by making the weaker assumption that "the scope for ... substitution is bounded" (Autor 2015).

The problem, however, is that the scope for substitution has proven not to be bounded in the way that the early task-based literature expected. Though accurately forecasting the capabilities of systems and machines is very difficult, that initial literature tended to underestimate them. For instance, the majority of tasks that were explicitly identified in Autor, Levy, and Murnane (2003) as being out of reach of machines can increasingly now be automated.[2] These papers tended to rely on a particular understanding of how systems and machines operate and

[1]For instance, in a canonical model with low-skill labour L^L and high-skill labour L, the production function is CES:

$$Y = \left[(A^L L^L)^{\frac{\sigma-1}{\sigma}} + (A^H L^H)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

By construction $\frac{\partial w^L}{\partial A^H} > 0$, $\frac{\partial w^L}{\partial A^L} > 0$ – and similarly for w^H.

[2]The paper noted that the "driving a truck" could not be readily automated, but a type of driverless vehicle appeared two years later; and it argued that "legal writing" and "medical diagnosis" could not be readily automated, yet document automation systems are now widespread in most major legal practices and there are a variety of technologies that can diagnose health problems. Also consider that Autor and Dorn (2013a) noted that order-taking and table-waiting could not be readily automated, but later that year the US restaurants Chili's and Applebee's announced they were installing 100,000 tablets to allow customers to order and pay without a human waiter;[3] Autor (2015) noted that the task of identifying a species of bird based on a fleeting glimpse could not be readily automated, but later that year an app was released to do that as well.[4].

the limits to their capabilities that this implied – known as the 'ALM hypothesis'.[5] Under this hypothesis, these tasks were believed to be out of reach of automation because they were 'non-routine' rather than 'routine'. But in some very significant cases this has proven not to be the case.

A new empirical literature has emerged that attempts to revise and update the boundaries to system and machine capabilities. For instance: Caines et al. (2017) argue that a task which is more 'complex' is harder to automate; Brynjolfsson and Mitchell (2017) and Brynjolfsson, Mitchell, and Rock (2018) argue that, among other factors, a task where the goal is easy to define, where it is straightforward to see if the goal has been achieved, and where there is lots of data, is easier to automate; and Frey and Osborne (2017) use machine learning techniques to identify a vector of possible signals that determine whether a task can be automated. Yet the fundamental difficulty with trying to identify new limits at all is that, in time, as with the ALM hypothesis, they are likely to change as well.

In this paper, I take an alternative approach. Rather than try and identify new limits, I start instead from a simple claim – though we cannot know exactly what systems and machines will be able to do in the future, we can be confident that they will be able to perform *more* types of tasks than they can today. This claim has also been made implicitly in the most recent literature (for instance in Autor and Acemoglu 2011, or Acemoglu and Restrepo 2018, where there are no explicit limits to the capabilities of machines). I call this, 'task encroachment'. And in what follows, I build a new model to explore the consequences of this process for the labour market, where the set of types of tasks in which labour has the comparative advantage over machines is gradually eroded away.

In this paper, I use the term 'capital' to refer to the wide variety of systems and machines used in the economy. In what follows, I introduce a new distinction between two types of capital – 'substituting' capital ('s-capital') and 'complementing' capital ('c-capital'). Here, c-capital is a q-complement to a distinct set of types of tasks, and an increase in its quantity or productivity raises the value of those complemented tasks. Under the early ALM hypothesis, these complemented tasks were 'non-routine' tasks; under the various newer attempts to identify the new boundaries to system and machine capabilities mentioned before, these tasks have a range of alternative properties.[6] However, in this paper, these comple-

[5]'ALM', after 'Autor, Levy and Murnane' – the authors of Autor, Levy, and Murnane (2003).

[6]The definition of 'q-complementarity' is more nuanced in a many-good setting, as in this paper. This is because in the set-up in which q-complementarity was originally defined – Hicks (1970), and Sato and Koizumi (1973) – the models had only a unique final good. When I use the term 'q- complement' I mean that an increase in the allocation of a factor's task input to

mented tasks are not indefinitely performed by labour – they are performed either by labour *or* by s-capital. This implies that labour benefits indirectly from technological change, only if it remains best-placed to perform those complemented tasks. And in this model, an increase in the quantity or productivity of s-capital erodes that comparative advantage of labour in performing these complemented tasks. This is task encroachment at work.

This distinction between two different types of capital, and their different effects on labour, is new and matters for our understanding of the impact of new technologies on labour.[7] The model in this paper is closely related to a set of existing task-based models that also study these effects (for instance, Zeira 1998, Acemoglu and Autor 2011, Hémous and Olsen 2016, Acemoglu and Restrepo 2018a, b). Yet because of the structure of production in these other models, and their use of only one type of capital, these two different effects on labour are entangled. The new model in this paper disentangles them in a revealing way. Each type of capital has only one effect – it either substitutes for labour, or q-complements it – and, as a result, it is possible to explore what happens if the set of tasks in which labour is q-complemented by c-capital shrinks, holding constant the 'intensity' of that q-complementarity.[8]

The new model in this paper, and the process of task encroachment that it captures, supports a more pessimistic view about the threat of automation. As s-capital becomes more productive, labour is forced to specialise in a shrinking set of types of complemented tasks. In a static version of the model, an increase in the quantity or productivity of s-capital drives down relative wages and the labour share of income and forces labour to specialise in a shrinking set of tasks. In a dynamic version, the endogenous accumulation of s-capital drives labour out the economy at an endogenously determined rate, and absolute wages fall towards zero. In the limit, labour is fully immiserated and 'technological unemployment' follows. As Autor and Salomons (2017a,b) put it, in their discussion of this new model, labour has "no place left to hide".

This paper is closely related to two recent papers, and the new distinction I

the production of a given good, ceteris paribus, causes the marginal product of the other task input involved in the production of that good to increase.

[7]Beyond the task-based literature, there are papers that involve two capital stocks. For instance, Steigum (2011) builds a growth model with two capital stocks, one of which is 'robot' capital, and more recent papers that have build on this approach: for instance, DeCanio (2016), Antony and Klarl (2019), Lankisch et al. (2019) and Antony and Klarl (2020).

[8]A notable exception to this is Berg, Buffie, and Zanna (2018). In this paper, there are two types of capital – 'robot' capital and 'traditional' capital. However, this model does not have a continuum of tasks and so cannot explore the process of task encroachment.

introduce between the two types of capital brings their conclusions together in a revealing way. The first is Acemoglu and Restrepo (2018a). In that paper, a similar process of task encroachment is at work, but labour is protected from immiseration by the endogenous creation of new tasks in which labour has the comparative advantage over capital. The second is Caselli and Manning (2019). In that paper, the authors show that in models where capital is perfectly elastically supplied and labour is the only fixed factor, then real wages must rise with technological progress. The new model in this paper starts from a similar point to Acemoglu and Restrepo (2018a) – but by changing one condition, that sufficient new tasks are created in which labour has the comparative advantage, the results change drastically, and the immiseration of labour follows.[9] In turn, the new distinction I introduce between s-capital and c-capital reveals that in a setting like this, where insufficient new tasks are created, c-capital takes on a central role in determining the direction of real wages – for if only s-capital is accumulated, as initially in this model, then labour is no longer the only fixed factor as in Caselli and Manning (2019), and real wages will *fall* with technological progress.[10]

This paper therefore identifies c-capital as an important new countervailing force against the immiseration of labour – particularly if insufficient new tasks are created in which labour has the comparative advantage. In turn, this provides additional theoretical support to the growing interest in policy interventions that encourage the development of "the 'right' kind of AI" (Acemoglu and Restrepo 2019b) and to an increasingly common demand that technologists develop machines that "work to augment humans, not simply replace us" (Brynjolfsson 2018) – or, in the language of this paper, to develop c-capital rather than s-capital.

1. A Static Model

In the new model that follows there are two sets of types of tasks. The first is a set of tasks that are performed only by 'complementing capital' ('c-capital'). This type of capital cannot perform the same type of tasks as labour. The second is a set of tasks that are either performed by labour or 'substituting capital' ('s-capital'). These tasks that are performed by either labour or s-capital are ordered in a line

[9]The model in this paper therefore could be thought of as a detailed account of a critical possible case that is briefly mentioned in Acemoglu and Restrepo (2018a) – what they call the 'horse equilibrium', where insufficient new tasks are created for labour to do (and workers, like horses in the past, are immiserated).

[10]As Caselli and Manning (2019) put it, in "the models of ... Susskind (2017), in which workers are harmed by new technology, rely on assuming ... that labor is not the only fixed factor".

from left to right, going from 'simple' to 'complex', and the relative productivity of labour with respect to s-capital increases as tasks become more 'complex'. This feature is similar to that in Acemoglu and Autor (2011).[11] To produce any good in the economy requires a combination of a task performed by c-capital and a task performed by either labour or s-capital.

The purpose of this new model is to study the effect of technological progress in the use of the different types of capital. In equilibrium there is a single cut-off and all of the tasks to the left of the cut-off are performed by s-capital with c-capital and all of the tasks to the right are performed by labour with c-capital. When capital is 'complementing', with a fixed and distinct role from labour in production, improvements in its capability have a neutral effect on labour – labour allocation across the task-continuum does not change, and wages relative to the return on capital remain the same. This is optimism at work. But when capital is 'substituting', increasingly capable s-capital erodes the set of types of tasks in which c-capital q-complements labour. The relative return to labour falls and labour is forced to specialise in a shrinking set of types of tasks. This is the new pessimism at work.

1.1. Consumers

There is a continuum of consumers $j \in [0, 1]$ who are either high-skilled workers or capitalists. If consumer j is a high-skilled worker he sells his labour l_j for a wage $w \geq 0$. If he is a capitalist with s-capital k_j^S or c-capital k_j^C he rents them to earn $r^S \geq 0$ and $r^C \geq 0$ respectively. There is a continuum of types of goods $x(i)$ where $i \in [0, 1]$ and each consumer j has Cobb-Douglas preferences over those goods:

$$\ln u(x_j) = \int_0^1 \theta(i) \ln x_j(i)\, di \tag{1}$$

Note that, given the Cobb-Douglas utility function, this economy can be captured by a representative consumer who owns all the factors. For simplicity, I assume that all goods have the same expenditure density:

ASSUMPTION 1: $\theta(i) = 1\ \forall i$.

[11] And in turn it shares features with Dornbusch, Fischer, and Samuelson (1977), from which Acemoglu and Autor (2011) also draw. Dornbusch, Fischer, and Samuelson (1977) is a two-country trade model where a continuum of goods are traded and the result is a Ricardian pattern of specialisation.

1.2. Production and Firms

Goods are produced by combining two different types of tasks, $z_1(i)$ and $z_2(i)$, where again $i \in [0,1]$. The first set of types of task, $z_1(i)$, are those that can performed by labour *and* s-capital. The second set, $z_2(i)$, can only be performed by c-capital. Perfectly competitive firms must hire factors to perform these tasks. The total stock of available factors is equal to the sum of l_j, k_j^S, k_j^C, owned by the consumers – L, K^S, and K^C respectively. The 'task-based production functions' for the goods are:

$$x(i) = z_1(i)^{\psi} z_2(i)^{1-\psi} \tag{2}$$

where $\psi \in (0,1)$. The 'factor-based production functions' for the tasks are:

$$\begin{aligned} z_1(i) &= a^S(i)K^S(i) + a^L(i)L(i) \\ z_2(i) &= a^C(i)K^C(i) \end{aligned} \tag{3}$$

where $L(i)$, $K^S(i)$, and $K^C(i)$ are the allocations of labour, s-capital, and c-capital to each type of task, and $a^L(i)$, $a^S(i)$ and $a^C(i)$ are their respective productivities. Again, these factor-based production functions for tasks reflect the fact that c-capital performs its own distinct set of tasks, but s-capital and labour perform the same tasks. The productivities of s-capital and labour combine to form a 'relative productivity schedule' over the $z_1(i)$ task-continuum:

$$A(i) = \frac{a^L(i)}{a^S(i)} \tag{4}$$

A second important assumption follows:

ASSUMPTION 2: *$A(i)$ is continuous, $A(0)$ is a positive constant, $A'(i) > 0$, and $A''(i) = 0$.*

The assumption that $A'(i) > 0$ is a 'comparative advantage' assumption.[12] It reflects two principles. First, as i increases, the task that is performed by labour or s-capital, $z_1(i)$, becomes more 'complex'. And secondly, that labour has an increasing comparative advantage over s-capital at performing more complex tasks. This is the sense in which labour is 'high' skilled. This reflects the fact that the most complex tasks draw on creative, problem-solving, and interpersonal faculties

[12] Acemoglu and Autor (2011) also use this approach.

of human beings that, as yet, are hardest to automate.[13]

1.3. Equilibrium

The Supply-Side

The firms must decide which factors to hire to perform the tasks that will produce each type of good. It is clear that to perform the tasks $z_2(i)$ the firms will rent c-capital – it is the only factor capable of performing those types of task. Less obviously, the firms will hire either labour *or* s-capital to carry out the tasks $z_1(i)$ – but never both together. This is Lemma 1:

LEMMA 1: *In equilibrium, there exists some cut-off $\tilde{i}$ such that s-capital works with c-capital to produce goods of type $i \in [0, \tilde{i}]$, and labour works with c-capital to produce the goods of type $i \in [\tilde{i}, 1]$.*

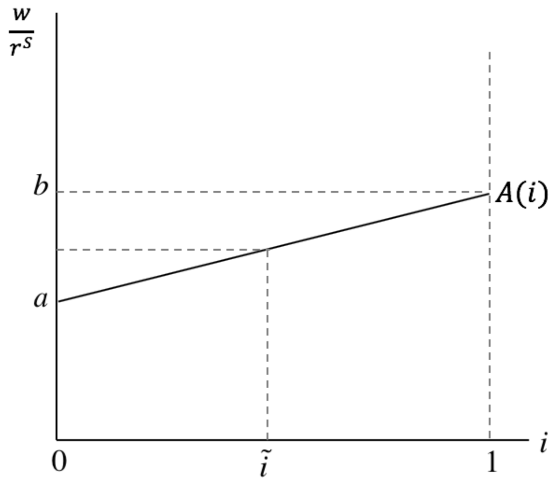
The proof is intuitive.[14] For any given w and r^S in equilibrium, a perfectly competitive firm will hire labour rather than s-capital to perform $z_1(i)$ if:

$$\frac{w}{a^L(i)} \leq \frac{r^S}{a^S(i)}$$
$$A(i) \geq \frac{w}{r^S} \tag{5}$$

which takes place to the right of $\tilde{i}$, given the properties of $A(i)$ in Assumption 1. The opposite argument applies for the choice of s-capital over labour. At $\tilde{i}$, the cost of performing a unit of $z_1(\tilde{i})$ to produce that good $\tilde{i}$ is the same for labour and s-capital. This is shown in Figure 1. For any $\frac{w}{r^S} \in [a, b]$ there is a single 'cut-off' type of good $\tilde{i}$. Given that factor-price ratio, firms would want to hire s-capital (to work with c-capital) to produce all goods to the left of $\tilde{i}$ and hire labour (to work with c-capital) to produce all goods to the right of $\tilde{i}$. This is also intuitive. $A(i)$ is a factor demand schedule, its shape determined by Assumption 1, and it describes the factor-price ratio for each type of good i that would make a firm indifferent between using either labour or s-capital for a given $z_1(i)$. Moving left to right along the task-continuum, if a firm is to remain indifferent, the relative price of labour must rise. This is because the relative advantage of labour over s-capital at performing $z_1(i)$ increases.

[13]Following Susskind (2019), the most complex tasks are relatively hard to *routinise*.
[14]Lemma 1 and the proof are similar to Lemma 1 in Acemoglu and Autor (2011).

Figure 1: The Relative Productivity Schedule



Given this reasoning and Lemma 1, (2) and (3) combine to form the following factor-based production functions for goods:

$$(6)\quad \begin{aligned} x(i) &= \left[a^S(i)K^S(i)\right]^{\psi}\left[a^C(i)K^C(i)\right]^{1-\psi} \quad \forall i \in [0, \tilde{i}] \\ x(i) &= \left[a^L(i)L(i))\right]^{\psi}\left[a^C(i)K^C(i)\right]^{1-\psi} \quad \forall i \in [\tilde{i}, 1] \end{aligned}$$

The Demand-Side

Call $\gamma(\tilde{i})$ the share of total consumer expenditure on all goods that are produced by s-capital and c-capital i.e. type $i \in [0, \tilde{i}]$. Assumption 1 implies that:

$$(7)\quad \begin{aligned} \gamma(\tilde{i}) &= \int_0^{\tilde{i}} \theta(i)\, di \\ &= \tilde{i} \end{aligned}$$

In turn, it follows from (2) that the share of total consumer expenditure that is spent on all the *tasks* performed by s-capital is equal to $\psi \cdot \tilde{i}$.[15] The same argument applies to the share of total consumer expenditure on tasks performed by labour and c-capital, equal to $\psi \cdot (1 - \tilde{i})$ and $(1 - \psi)$.

Equilibrium Factor Prices and Specialisation

[15]This is because the task-based production function for goods is Cobb-Douglas. To see this formally, call the implicit 'price' of $z_1(i)$, $p_{z_1}(i)$. Perfectly competitive firms will set this 'price' equal to the marginal revenue product of $z_1(i)$ in producing $x(i)$, which is $p(i) \cdot \psi z_1(i)^{\psi-1} z_2(i)^{1-\psi}$. As a result the $z_1(i)$ share of the expenditure on a particular $x(i)$ is $\frac{p(i)\cdot\psi\cdot z_1(i)^{\psi-1} z_2(i)^{1-\psi}\cdot z_1(i)}{p(i)\cdot x(i)} = \psi$.

As firms are perfectly competitive, the zero-profit condition requires that the total consumer expenditure on the tasks that are performed by each factor is equal to the income that the factor receives in return for carrying out those tasks. As a result, given (7), the following conditions must hold:

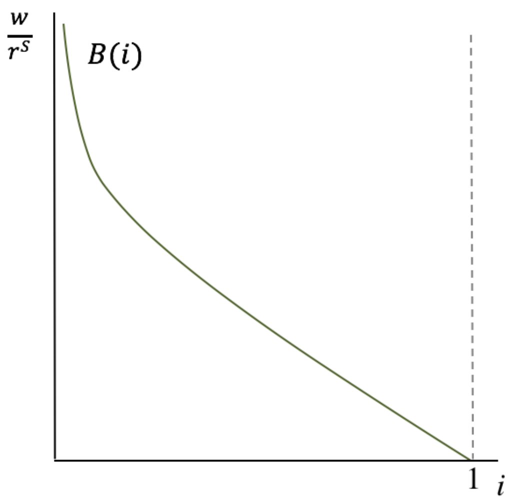
$$\begin{aligned} r^S K^S &= \psi \cdot \tilde{i} \cdot Y \\ wL &= \psi \cdot (1-\tilde{i}) \cdot Y \\ r^C K^C &= (1-\psi) \cdot Y \end{aligned} \tag{8}$$

where $Y = r^S K^S + wL + r^C K^C$. The first two expressions in (8) imply:

$$\begin{aligned} \frac{w}{r^S} &= \frac{1-\tilde{i}}{\tilde{i}} \cdot \frac{K^S}{L} \\ &= B(\tilde{i}) \end{aligned} \tag{9}$$

$B(i)$ generates a market equilibrium schedule. This is shown in Figure 2.

Figure 2: The Zero-Profit Schedule



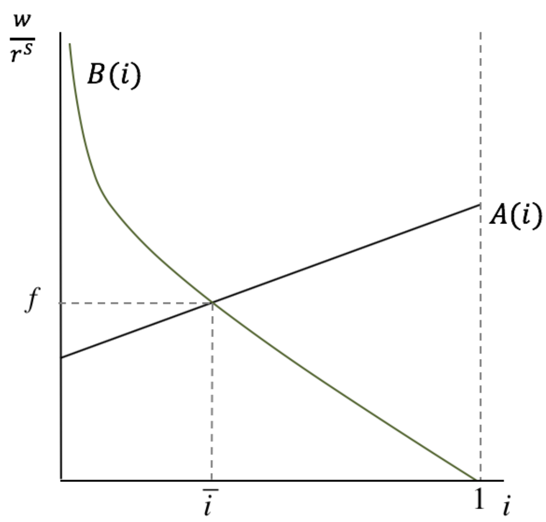
From inspection, it is clear that for any $\frac{w}{r^S}$ there is a unique cut-off good of type $\tilde{i}$ that ensures market equilibrium holds. (9) and Assumption 1 imply: $B'(i) < 0$; $B''(i) > 0$; $\lim_{i \to 0} B(i) = \infty$; and $B(1) = 0$.

The $A(i)$ schedule describes the pattern of specialisation $\tilde{i}$ for each factor-price ratio $\frac{w}{r^S}$. This is a factor-demand schedule. $B(i)$ describes the factor-price ratio $\frac{w}{r^S}$ that ensures market equilibrium for each pattern of specialisation $\tilde{i}$. This is a zero profit schedule. To derive the equilibrium cut-off $\bar{i}$, rather than a hypothetical cut-off $\tilde{i}$, these two schedules are combined. This is shown in Figure 3 and Proposition 1 follows.

PROPOSITION 1: *Given the properties of $A(i)$ and $B(i)$, there is a unique equilibrium factor-price ratio, f, and a unique equilibrium cut-off type of good, $\bar{i}$.*

Uniqueness can be seen from an inspection of Figure 3. Given the properties of the $A(i)$ and $B(i)$ schedules in Assumption 2, the $A(i)$ schedule must start below the $B(i)$ schedule and the schedules must cross only once.

Figure 3: Market Equilibrium



In equilibrium there is a clear pattern of specialisation. It is 'Ricardian'.[16] s-capital specialises in producing the types of goods $i \in [0, \bar{i}]$ with c-capital, and labour specialises in producing the goods $i \in [\bar{i}, 1]$ with c-capital – i.e. the hypothetical cut-off $\tilde{i}$ is replaced by the actual cut-off $\bar{i}$ in (6). The full set of relative factor prices follow from (8) and (9) – again, recognising that in equilibrium the actual cut-off $\bar{i}$ replaces the hypothetical cut-off $\tilde{i}$:

$$\begin{aligned} \frac{w}{r^S} &= \frac{1-\bar{i}}{\bar{i}} \cdot \frac{K^S}{L} \\ \frac{w}{r^C} &= \frac{\psi \cdot (1-\bar{i})}{1-\psi} \cdot \frac{K^C}{L} \\ \frac{r^S}{r^C} &= \frac{\psi \cdot \bar{i}}{1-\psi} \cdot \frac{K^C}{K^S} \end{aligned} \tag{10}$$

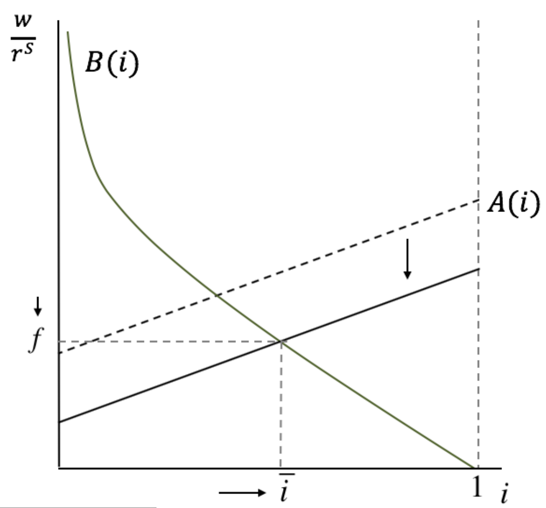
[16]Equilibrium here is similar to Dornbusch, Fischer, and Samuelson (1977), though theirs is an equilibrium in international trade.

1.4. Comparative Statics

This model can be used to compare the effect of technological progress in the use of the two different types of capital. First consider c-capital, and a rise in $a^C(i)$ across the task-continuum. This has no effect on the wage relative to either the return on s-capital, $\frac{w}{r^S}$, or c-capital, $\frac{w}{r^C}$, nor on the pattern of labour specialisation, $\bar{i}$. This follows from the definition of the $A(i)$ and $B(i)$ schedules and (10). Figure 3 remains unchanged. This is because c-capital is a q-complement to both labour and s-capital in the production of all types of goods and, because the task-based production functions for goods in (2) are Cobb-Douglas, this implies a rise in the marginal productivity of c-capital in producing a given good causes an equiproportionate rise in the marginal productivity of either the labour or s-capital producing that good.[17] This is the traditional channel of optimism in the task-based literature – there exists a large set of types of tasks out of reach of automation, in which labour is complemented by capital.

However, s-capital does have an effect on both relative wages. A uniform rise in the relative productivity of s-capital is shown in Figure 4. The result is a fall in $\frac{w}{r^S}$ and $\frac{w}{r^C}$ and a rise in $\bar{i}$ – labour is forced to specialise in a shrinking set of types of tasks, and is relatively worse off.[18] This is because s-capital is a perfect substitute for labour in performing those tasks that are q-complemented by c-capital. labour's comparative advantage diminishes, and is forced to specialise in a shrinking set of q-complemented tasks. This is the new pessimism at work.

Figure 4: A New Market Equilibrium



[17]Note again in this many-good setting the traditional definition of 'q-complementarity' does not apply straightforwardly. This is because in those settings there is only a unique final good.

[18]Exploring absolute, rather than relative, prices is more complex are requires the simulations that follow in the dynamic setting.

2. A Dynamic Model

In the static model, as s-capital becomes more productive, consumers do not respond to any changes in the rate of return to capital. Now I place that new model in a dynamic setting and introduce an endogenous process of s-capital accumulation. The outcome is remorselessly pessimistic – labour is displaced at an endogenously determined rate, is forced into specialising in a shrinking set of tasks, and absolute wages are driven to zero. No steady-state is possible until labour has been entirely driven out the economy by s-capital i.e. the economy must approach a steady-state where $\bar{i}(t) = 1$. labour is fully immiserated, and technological unemployment follows.

To solve this dynamic model, I nest the static analysis in the previous section in a traditional Ramsey growth model framework. In any given t, the factor prices and pattern of specialisation are determined instantaneously by that static analysis – the $A(\cdot)$ and $B(\cdot)$ schedules in (5) and (9) are now time dependent such that $A(i,t)$ depends upon $a^S(i,t)$ and $a^L(i,t)$, and $B(i,t)$ on $\bar{i}(t)$, $K^S(t)$, and $K^C(t)$. A further important feature of this model is the innovative use of a numeraire price normalisation using a numeraire good. The price of this numeraire good is set to one, and the prices of all other goods are relative to this good. This numeraire price normalisation makes the dynamic model far more tractable, in two ways. First it allows me to reduce the dimensionality of the many-good model. As I will show, with Cobb-Douglas preferences across the range of goods, the law of motion for the numeraire good is the same as the law of motion for aggregate consumption across all goods – and so deriving the law of motion for the numeraire good allows me to focus on aggregate consumption alone, rather than track the full set of laws of motion for each good. Secondly, with Cobb-Douglas production across the range of goods, the numeraire good allows me to derive analytically tractable expressions for the *absolute* factor prices, r^C and r^S. These were not required in the static analysis in the previous section – static equilibrium could be characterised by the *relative* factor prices alone, as in (10). However, identifying the dynamic equilibrium requires the absolute factor prices as well.[19]

[19] It is also possible to use a simplex price normalisation to solve the dynamic model, where $\int_0^1 p(i)di = 1$. However, this approach can only be solved computationally and is far less tractable than the numeraire price normalisation that I use.

2.1. Consumers

Consumers have the same preferences as before. Again, the economy can be captured by a representative consumer. Only s-capital $K^S(t)$ is accumulated, and the consumer faces the dynamic maximisation problem:

$$
\begin{aligned}
&\max_{\boldsymbol{x(t)}} \int_0^\infty e^{-\rho t} \int_0^1 \ln x(i,t)\, di\, dt \\
&\text{s.t.} \\
&\dot{K}^S(t) = r^S(t) \cdot K^S(t) + r^C(t) \cdot K^C + w(t) \cdot L - c(t) \\
&K^S(0) = K_0^S \\
&K^S(t) \geq 0
\end{aligned} \tag{11}
$$

I assume there is exogenous growth in the productivity of s-capital, but no growth in the productivity of c-capital. Any growth process must satisfy the following:

ASSUMPTION 3: *For any exogenous growth process used, it must be the case that* $a^S(\tilde{\tilde{i}}, t) \leq a^S(\tilde{i}, t) \cdot \frac{a^L(\tilde{\tilde{i}}, t)}{a^L(\tilde{i}, t)}$ *for* $\tilde{\tilde{i}} > \tilde{i}\ \forall t$.

This is the dynamic version of Assumption 2. It ensures that, as technological progress takes place, s-capital does not become so productive in more complex tasks (i.e. those with a higher i) as to overturn the general principle that labour has the comparative advantage in these more complex tasks i.e. that $A_1(i,t) \geq 0\ \forall t$.[20] Initially, I assume that the particular growth process is:

$$
\frac{\dot{a}^S(i,t)}{a^S(i,t)} = g \quad \forall i, t \tag{12}
$$

In the Appendix I show that this satisfies Assumption 3.

2.2. Production and Firms

Production is the same as in the static setting – the task-based production function for goods, and the factor-based production functions for tasks, are as in (2) and (3). For simplicity, I assume in the dynamic setting that there is no depreciation. In closing this section, I explain why depreciation does not change the central results.

[20]Where $A_1(i,t)$ is the derivative of $A(i,t)$ with respect to i.

The traditional approach to finding the steady-state in a Ramsey growth model with technological progress like this is to re-define the variables in 'effective' terms, dividing each variable by the prevailing level of labour-augmenting technology. The result is that a steady-state is reached not in the actual variables, but instead in these 'effective' variables, the variable 'per efficiency unit of labour'. In exactly the same way, solving this model requires that the s-capital augmenting technological progress I am considering is instead exactly reflected in a process of c-capital-augmenting technological progress. Consider again a good $x(\tilde{i},t)$ that is produced by s-capital and c-capital. The transformation of the production function is as follows:

$$
\begin{aligned}
x(\tilde{i},t) &= \left[a^S(\tilde{i},t)\cdot K^S(\tilde{i},t)\right]^{\psi}\left[a^C(\tilde{i},t)\cdot K^C(\tilde{i},t)\right]^{1-\psi} \\
&= \left[K^S(\tilde{i},t)\right]^{\psi}\left[a^S(\tilde{i},t)^{\frac{\psi}{1-\psi}}\cdot a^C(\tilde{i},t)\cdot K^C(\tilde{i},t)\right]^{1-\psi}
\end{aligned} \tag{13}
$$

(13) implies that a c-capital augmenting process of technological change in $a^S(\tilde{i},t)^{\frac{\psi}{1-\psi}}$ is identical to the s-capital augmenting process of technological change in $a^S(\tilde{i},t)$ in Assumption 3. This transformation has an important role in solving the dynamic model.

2.3. Dynamic Equilibrium

From the maximisation problem in (11) a current-value Hamiltonian follows:

$$
\begin{aligned}
H = &\int_0^1 \ln x(i,t)\, di \\
&+ \mu(t)\left[r^C(t)\cdot K^C + r^S(t)\cdot K^S(t) + w(t)\cdot L - \int_0^1 x(i,t)\cdot p(i,t)\, di\right]
\end{aligned} \tag{14}
$$

It is possible to solve this Hamiltonian – with one co-state variable $\mu(t)$ and a continuum of control variables, $x(i,t)$ where $i \in [0,1]$ – in the traditional way. I confirm this in the Appendix. A set of first order conditions follow for each good $x(i,t)$:

$$
\begin{aligned}
H_{x(i)} &= \frac{1}{x(i,t)} - \mu(t)\cdot p(i,t) \\
&= 0
\end{aligned} \tag{15}
$$

And for $K^S(t)$, the state variable:

$$H_{K^S} = \mu(t) \cdot r^S(t) \\ = \rho \cdot \mu(t) - \dot{\mu}(t) \tag{16}$$

Together, (15) and (16) imply that for good $x(i,t)$:

$$\frac{\dot{x}(i,t)}{x(i,t)} = -\frac{\dot{p}(i,t)}{p(i,t)} + r^S(t) - \rho \tag{17}$$

This is derived in the Appendix. In a traditional Ramsey growth model, there is only a unique final output and so there is no need to consider how the price of the good changes over time. But (17) shows that in this many-good setting the rate of growth of demand for $x(i,t)$ will depend upon how its price changes over time. To maintain tractability, I now use a numeraire price normalisation. In particular, I assume that:

ASSUMPTION 4: *The numeraire good is $x(0,t)$ and so $p(0,t) = 1$ $\forall t$. Factor productivities and factor stocks are finite such that $\bar{i}(t) \neq 0$ $\forall t$.*

The price of good 0 is set to 1, and the prices of all other goods are in terms of that good. As I will show, this now makes the dynamic model far more tractable. Given Assumption 4 it follows from (17) that for $x(0,t)$:

$$\frac{\dot{x}(0,t)}{x(0,t)} = r^S(t) - \rho \tag{18}$$

Given Assumption 1 it follows that the law of motion for the numeraire good $x(0,t)$ is the same as the law of motion for total consumption $c(t)$.[21] And so:

$$\frac{\dot{c}(t)}{c(t)} = r^S(t) - \rho \tag{19}$$

It follows that if $x(\tilde{i},t)$ reaches a steady-state then total consumption $c(t)$ will also

[21] Where $c(t) = \int_0^1 x(i,t) \cdot p(i,t)\, di$. To see this note that since $p(0,t) = 1$:

$$c(t) = \frac{x(0,t) \cdot p(0,t)}{\theta(0)} = \frac{x(0,t)}{\theta(0)}$$

and so $\frac{\dot{c}(t)}{c(t)} = \frac{\dot{x}(0,t)}{x(0,t)}$ since $\theta(0)$ is constant and equal to 1.

be in steady-state. From now, I use the law of motion in (19) and focus on $c(t)$. The law of motion for $K^S(t)$ follows from (11):

$$\dot{K}^S(t) = Y(t) - c(t) \tag{20}$$

At this point, the dynamic system is expressed in terms of $c(t)$ and $K^S(t)$. In order to find the steady-state in this model it is necessary to transform these variables into 'effective' terms, as with a traditional Ramsey growth model. For any variable $v(t)$, I use the following transformations:

$$\hat{v}(t) = \frac{v(t)}{a^S(0,t)^{\frac{\psi}{1-\psi}}} \qquad \hat{\hat{v}}(t) = \frac{v(t)}{\bar{i}(t) \cdot a^S(0,t)^{\frac{\psi}{1-\psi}}} \tag{21}$$

These transformations are new and differ from those used in any traditional Ramsey growth model. The intuition for the form of these effective variable is revealed once the dynamic equilibrium is derived. But from inspection it is clear that a '$\hat{...}$' term is 'effective' with respect to a term that captures the productivity of s-capital at time t (reflecting the transformation in (13)), whereas the '$\hat{\hat{...}}$' term is 'effective' with respect to a term that captures the productivity of s-capital *and* the value of the cut-off at any given moment in time, $\bar{i}(t)$. In the analysis that follows I look for an equilibrium in $(\hat{c}, \hat{\hat{K}}^S)$ space i.e. in 'effective' consumption and s-capital space.

First, consider $\hat{c}(t)$. (19) and (21) imply a law of motion for $\hat{c}(t)$:

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = r^S(t) - \rho - \frac{\psi}{1-\psi} \cdot g \tag{22}$$

Secondly, consider $\hat{\hat{K}}^S(t)$. (20) and (21) imply a law of motion for $\hat{\hat{K}}^S(t)$:

$$\frac{\dot{\hat{\hat{K}}}^S(t)}{\hat{\hat{K}}^S(t)} = \frac{\hat{Y}(t) - \hat{c}(t)}{\hat{K}^S(t)} - \left[g^{\bar{i}}(t) + \frac{\psi}{1-\psi} \cdot g\right] \tag{23}$$

where $g^{\bar{i}}(t)$ is the growth rate in $\bar{i}(t)$. Both the laws of motion in (22) and (23) can be expressed in terms of $\hat{c}(t)$ and $\hat{\hat{K}}^S(t)$ alone. This is what is required to study dynamic equilibrium in $\hat{c}(t), \hat{\hat{K}}^S(t))$ space. To re-write (22) in this way requires an expression for $r^S(t)$ in terms of $(\hat{\hat{K}}^S(t)$; to re-write (23) in this way requires an expression for $\hat{Y}(t)$ and $g^{\bar{i}}(t)$ in terms of $\hat{\hat{K}}^S(t)$. The expressions for $r^S(t)$ and $\hat{Y}(t)$ in terms of $\hat{\hat{K}}^S(t)$ are derived in the Appendix and shown to be equal to:

(24) $$r^S(t) = \hat{\hat{K}}^S(t)^{\psi-1} \cdot D$$

and:

(25) $$\hat{Y}(t) = \hat{\hat{K}}^S(t)^{\psi} \cdot \frac{D}{\psi}$$

where D is a positive constant equal to $\left(a^C(0) \cdot K^C\right)^{1-\psi} \cdot \psi$.[22] The expression for $g^{\bar{i}}(t)$ in terms of $\hat{\hat{K}}^S(t)$ follows by definition:

(26) $$g^{\bar{i}}(t) = \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^S(t)} \cdot \dot{\hat{\hat{K}}}^S(t) \cdot \frac{1}{\bar{i}(t)}$$

Using the expression for $r^S(t)$ in (24), $\hat{Y}(t)$ in (25), and $g^{\bar{i}}(t)$ in (26), the laws of motion in (22) and (23) can now be re-expressed in terms of $\hat{c}(t)$ and $\hat{\hat{K}}^S(t)$ alone as required. The law of motion for $\hat{c}(t)$ follows from (22) and (24):

(27) $$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \hat{\hat{K}}^S(t)^{\psi-1} \cdot D - \rho - \frac{\psi}{1-\psi} \cdot g$$

The law of motion for $\hat{\hat{K}}^S(t)$ in (23) is more complex to derive. The full derivation

[22](25) implies that the *level* of $r^S(t)$ depends on the choice of the numeraire good, $\tilde{i}$ – if a different $\tilde{i}$ is chosen, the level of $a^S(\tilde{i}, t)$ and $a^C(\tilde{i}, t)$ will differ from those in (25) where $\tilde{i} = 0$. However, this does not affect the important features of absolute factor prices in equilibrium. In the case where there is a one-off change in the productivity of s-capital, so long as that change is uniform – as in Dornbusch, Fischer, and Samuelson (1977) – (25) implies that, *ceteris paribus*, $r^S(t)$ will always move in the same direction regardless of the choice of numeraire. In the case where there is an increase in the growth rate of the productivity of s-capital, so long as the growth rates remain the same across different tasks, *ceteris paribus*, (25) implies that $r^S(t)$ will always increase at the same rate regardless of the choice of numeraire. To explore non-uniform changes in productivity, or to make the *level* of $r^S(t)$ invariant to the normalisation, it is necessary to use a simplex price normalisation. But this leads to an implicit, rather than explicit and tractable, solution to the model. This need for uniformity is an interesting limitation of Dornbusch, Fischer, and Samuelson (1977) that was not explored. Complications involving price normalisations are discussed elsewhere in the literature – Dierker and Grodal (1999), for example, on models of imperfect competition.

is shown in the Appendix. Using (25) and (26), it can be written as:

$$\frac{\dot{\hat{\hat{K}}}^S(t)}{\hat{\hat{K}}^S(t)} = \frac{\hat{\hat{K}}^S(t)^{\psi-1} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\hat{c}(t)}{\hat{\hat{K}}^S(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^S(t)} \cdot \hat{\hat{K}}^S(t)} \tag{28}$$

With a traditional Ramsey growth model, the standard approach to identifying steady-state is to consider these schedules in $(\hat{c}, \hat{\hat{K}}^S)$ space by plotting their stable arms. (27) implies that the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule, the stable arm for $\hat{c}(t)$, is:

$$\hat{\hat{K}}^{S*} = \left[\left[\rho + \frac{\psi}{1-\psi} \cdot g\right] \cdot \frac{1}{D}\right]^{\frac{1}{\psi-1}} \tag{29}$$

This expression has an identical form to the stable arm in a traditional Ramsey growth model with a labour-augmenting growth process taking place at rate $\frac{\psi}{1-\psi} \cdot g$ – the stable arm for $\hat{c}(t)$ is simply a vertical schedule at some fixed $\hat{\hat{K}}^{S*}$ that ensures $r^S(t) = \rho + \frac{\psi}{1-\psi} \cdot g$ and $\hat{c}(t)$ is constant. (28) implies that the stable arm for $\hat{\hat{K}}^S(t)$, the $\dot{\hat{\hat{K}}}^S(t) = 0$ schedule, is equal to:

$$\hat{c}(t) = \hat{\hat{K}}^S(t)^{\psi} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g \cdot \hat{\hat{K}}^S(t) \tag{30}$$

This is almost identical to the stable arm in a traditional Ramsey growth model, except in one important respect – the presence of $\bar{i}(t)$. The reason that $\bar{i}(t)$ appears in the $\dot{\hat{\hat{K}}}^S(t) = 0$ schedule in this model is critical. This is because in any period t, s-capital is only used to produce $\bar{i}(t)$ of the goods in the economy. The remaining $1 - \bar{i}(t)$ goods are produced by labour whose productivity is unaffected by technological progress (reflected, as in Assumption 3, by improvements in the productivity of s-capital). However as $\bar{i}(t)$ rises, and more goods are produced by s-capital rather than labour, the production of more goods in the economy is affected by that technological progress. As $\bar{i}(t)$ rises it is as if the 'effective' rate of technological progress – this is $\bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g$ in (30) – rises. Indeed, the increase in $\bar{i}(t)$ in this new model with many goods has the same consequence as an increase in technological progress in a traditional Ramsey growth model with a unique final good.[23] Intuitively, in a traditional Ramsey growth model with a

[23]If the rate of technological progress were to increase a traditional Ramsey growth model,

unique final good, the economy 'feels the full force' of the technological progress, whereas in this many-good model only $\bar{i}(t)$ of the economy does (i.e. the $\bar{i}(t)$ of the economy that is exposed to s-capital).

$\bar{i}(t)$ also has an important role in determining $r^S(t)$ in (24). That expression implies that $r^S(t)$ is decreasing in $\hat{\hat{K}}^S(t)$. Given the definition of $\hat{\hat{K}}^S(t)$ implied by (21), this means that $r^S(t)$ is decreasing in $K^S(t)$ and increasing in $a^S(0,t)$ – as in a traditional Ramsey growth model – but that it is *also* increasing in $\bar{i}(t)$. This is because while there are diminishing returns to the use of $K^S(t)$ in the production of any given good, increasing the range of goods $\bar{i}(t)$ that $K^S(t)$ is used to produce 'spreads out' the stock of s-capital across the economy, and offsets those diminishing returns. It is this new role for $\bar{i}(t)$ that means the transformations in (21) must be used to find the dynamic equilibrium.

Because of the $\bar{i}(t)$ term in the stable arm for $\hat{\hat{K}}^S(t)$ in (30), dynamic equilibrium is not analytically tractable. It requires a non-linear simulation of the three equation system:[24]

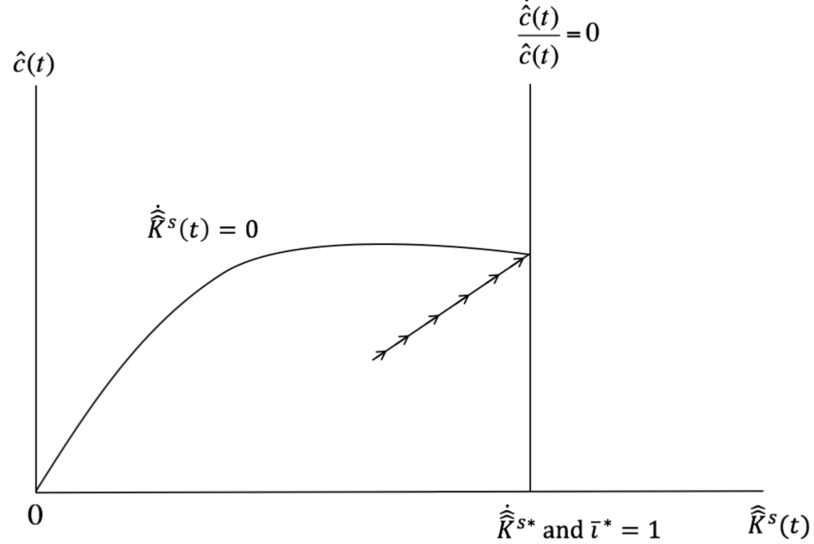
(31)
$$\begin{aligned}
\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \hat{\hat{K}}^S(t)^{\psi-1} \cdot D - \rho - \frac{\psi}{1-\psi} \cdot g \\
\frac{\dot{\hat{\hat{K}}}^S(t)}{\hat{\hat{K}}^S(t)} &= \frac{\hat{\hat{K}}^S(t)^{\psi-1} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\hat{c}(t)}{\hat{\hat{K}}^S(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^S(t)} \cdot \hat{\hat{K}}^S(t)} \\
\bar{i}(t) &= f(K^S(t), A(\bar{i}(t), t), L)
\end{aligned}$$

The first two are the familiar differential equations for $\hat{c}(t)$ and $\hat{\hat{K}}^S(t)$. The third is a static equation that determines $\bar{i}(t)$ in any given t. However, despite this complexity, it is still possible to identify analytically the unique steady-state that this model approaches and also to provide an intuitive conjecture for transition to this steady-state, without performing the simulation. This is shown in Figure 5 and Proposition 2 follows.

the corresponding 'effective' capital schedule would fall, requiring a lower rate of effective consumption to ensure that the effective capital stock is constant.

[24] $\hat{c}(t)$ is a forward-looking state variable, $\hat{\hat{K}}^S(t)$ is a backward-looking state variable, and $\bar{i}(t)$ is a simultaneously endogenous non-state variable (i.e. it has no associated differential equation).

Figure 5: Dynamic Equilibrium



PROPOSITION 2: *When the static model with s-capital is placed in a dynamic setting with endogenous s-capital accumulation, the economy approaches a unique steady-state at* $\hat{c}^*$, $\hat{\hat{K}}^{S*}$, *where* $\bar{\imath}^* = 1$ *– in this steady-state, labour is entirely driven out by s-capital. During transition,* $w(t)$ *is driven to zero. The capital share of income rises steadily to 1.*

Given the restrictive modelling assumptions, Proposition 2 is best thought of as a possibility result – that it is possible, in a dynamic model with optimising agents in a general equilibrium setting, for wages and the labour share of income to decline to zero because increasingly capable s-capital displaces labour completely in production. The reasoning for Proposition 2 is as follows.

First, consider the steady-state itself. This exists where the stable arms for $\hat{c}(t)$ and $\hat{\hat{K}}^S(t)$ intersect. The steady-state is unique and must take place when $\bar{\imath}(t) = 1$ i.e. when labour is entirely driven out of the economy by s-capital. To see why, first note that the stable arm for $\hat{c}(t)$ in ($\hat{\hat{K}}^S$, $\hat{c}$) space is a vertical schedule at the level of $\hat{\hat{K}}^{S*}$ in (29). Now consider the following proof by contradiction. Suppose that the stable arm for $\hat{\hat{K}}^S(t)$ crosses the stable arm for $\hat{c}(t)$ where $\bar{\imath}(t) \neq 1$. In this potential steady-state the productivity of s-capital grows at a rate $\frac{\psi}{1-\psi} \cdot g$. If this is to be a steady-state, then $\bar{\imath}(t)$ must remain constant. If $\bar{\imath}(t)$ does not remain constant, then the stable arm for $\hat{\hat{K}}^S(t)$ will shift. This is implied by (30). The analysis of the static equilibrium in the previous section implies that $\bar{\imath}(t)$ will only stay constant if $K^S(t)$ decreases to offset this increase in productivity.

But if $K^S(t)$ decreases such that $\bar{i}(t)$ remains constant then $\hat{\hat{K}}^S(t)$ will fall, given its definition in (21), violating the condition that $\hat{\hat{K}}^{S*}$ is constant at the steady-state, as required by (29). $K^S(t)$ must therefore rise to ensure that $\hat{\hat{K}}^{S*}$ remains constant. However, this growth in the stock of s-capital increases $\bar{i}(t)$ further. From the definition of the stable arm for $\hat{\hat{K}}^S(t)$ in (30) this will drive down the intersection point of the stable arms for $\hat{c}(t)$ and $\hat{\hat{K}}^S(t)$. The steady-state value of $\hat{c}^*$ will be driven down. Repeating this argument implies that if steady-state is to exist in $\hat{\hat{K}}^S(t)$–$\hat{c}(t)$ space, it must take place when $\bar{i}(t) = 1$.[25]

Now consider the transition path to this steady-state, where $\bar{i}(t) = 1$. Figure 5 shows this path when the steady-state is approached from the left i.e. $\hat{\hat{K}}^S(0) < \hat{\hat{K}}^{S*}$. The transition path drawn is based on three observations. The first is that the stable arm for $\hat{c}(t)$ is known, given in (29). The second is that as $\hat{\hat{K}}^S(t)$ increases, $\bar{i}(t)$ also increases – this is intuitive, but proven in the Appendix. This gives the stable arm for $\hat{\hat{K}}^S(t)$ its hump-shape in Figure 5, given (30).[26] The third is that, supposing the economy reaches the steady-state where $\bar{i}(t) = 1$, the new model must collapse to have the same general form as a traditional Ramsey growth model in which there is labour and capital, and a labour-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$ – but rather than there being labour and capital, there is instead only two types of capital, with an s-capital-augmenting process of technological change taking place at rate g that is equivalent to a c-capital-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$. As a result, locally to the steady-state, the new model has the same approximated saddle-path as this traditional Ramsey growth model. This is what is shown in Figure 5. Note that, unlike the Ramsey model, this steady-state is only reached in the limit.[27]

This transition path is intuitive – during transition effective s-capital, $\hat{\hat{K}}^S(t)$, effective consumption, $\hat{c}(t)$, and the cut-off $\bar{i}(t)$ rise and approach the steady state. In effect, the accumulation of s-capital drives up consumption but also drives out labour. This process is driven by the fact that s-capital is accumulated not only to offset the fact that it is becoming more productive, as in transition to steady-state in a traditional Ramsey growth model, but also because s-capital is becoming

[25]Note that it is not possible for a steady-state where $\bar{i}(t) = 0$ – although $\bar{i}(t)$ is constant, this would imply there is no s-capital and so the steady-state condition in (29) is violated.

[26]In effect, the $\dot{\hat{\hat{K}}}^S(t) = 0$ schedule is bounded below by a hypothetical $\dot{\hat{\hat{K}}}^S(t) = 0$ schedule where $\bar{i}(t)$ is fixed at 1 and above by a hypothetical $\dot{\hat{\hat{K}}}^S(t) = 0$ schedule where $\bar{i}(t)$ is fixed at 0.

[27]By 'collapse', I mean the static three-factor analysis is replaced by a two-factor analysis where there is no labour, and s- and c-capital combine to produce all goods.

more intensively used across the economy – $\bar{i}(t)$ is rising, offsetting the diminishing returns to s-capital in the production of any particular good. It is as if the economy is chasing a steady-state that is continually slipping out of its grasp – until $\bar{i}(t) = 1$ and labour is fully driven out.

During transition, wages are driven to zero. To see this, first note from (24) that as $\hat{\hat{K}}^S(t)$ rises during transition, $r^S(t)$ must fall. In turn, consider that the relative wage of labour with respective to s-capital as steady-state is approached is equal to:

$$\begin{aligned} \lim_{t\to\infty} \frac{w(t)}{r^S(t)} &= \lim_{t\to\infty} A(\bar{i}(t), t) \\ &= \lim_{t\to\infty} \frac{a^L(\bar{i}(t), t)}{a^S(\bar{i}(t), t)} \end{aligned} \tag{32}$$

and since $\lim\limits_{t\to\infty} \bar{i}(t) = 1$ it follows that:

$$\begin{aligned} \lim_{t\to\infty} \frac{a^L(\bar{i}(t), t)}{a^S(\bar{i}(t), t)} &= \lim_{t\to\infty} \frac{a^L(1, t)}{a^S(1, t)} \\ &= 0 \end{aligned} \tag{33}$$

given that $a^L(i, t)$ is constant over time $\forall i$, and there is continuous growth in $a^S(i, t)\ \forall i$, as in (12), so $\lim\limits_{t\to\infty} a^S(1, t) = \infty$. This means that $\frac{w(t)}{r^S(t)}$ is driven to zero. And since $r^S(t)$ must fall during transition, this implies $w(t)$ is driven to zero as well. To ensure that the steady-state is well-defined, I assume that once wages are equal to zero, labour chooses not to work at all rather than to work for nothing at all.

Note that the particular transition path draw in Figure 5 is still a conjecture, and for an important reason – the approximated saddle-path in Figure 5 is only correct locally to the steady-state. It is based on the claim that $\bar{i}(t) = 1$ in steady-state, but as soon as $\hat{\hat{K}}^S(t)$ deviates from $\hat{\hat{K}}^{S*}$ along the saddle-path, $\bar{i}(t) \neq 1$, and this claim no longer holds. In order to find the actual saddle-path to this steady-state, it is necessary to perform a non-linear simulation. I show an example in the Appendix.

Proposition 2 is also derived without depreciation. This may seem like a significant omission, since the depreciation of the existing stock of s-capital may appear to act as a counterbalance to the accumulation of s-capital and the displacement of labour. But this intuition is incorrect. Suppose depreciation takes place at rate δ. As in a traditional Ramsey growth model, the introduction of depreciation changes the stable arm for $\hat{c}(t)$:

$$(34) \qquad \hat{\hat{K}}^{S*} = \left[\left[\rho + \delta + \frac{\psi}{1-\psi} \cdot g\right] \cdot \frac{1}{D}\right]^{\frac{1}{\psi-1}}$$

and also the stable arm for $\hat{\hat{K}}^S(t)$, $\dot{\hat{\hat{K}}}^S(t) = 0$:

$$(35) \qquad \hat{c}(t) = \hat{\hat{K}}^S(t)^{\psi} \cdot D - \bar{i}(t) \cdot \left[\delta + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{K}}^S(t)$$

The implication of (34) and (35) is that while the introduction of depreciation will change the level of steady-state $\hat{\hat{K}}^{S*}$ – implied by (34) – it is still the case that the model must approach a steady-state in $(\hat{c}, \hat{\hat{K}}^S)$ space when $\bar{i}(t) = 1$. This is again implied by (35) and the reasoning used to derive Proposition 2. The outcome again is remorselessly pessimistic for labour.

As a final observation note that, even in this representative agent setting, it is possible to see that the outcome for the owners of capital is remorselessly *optimistic* – in contrast to that outcome for labour. As $\bar{i}(t)$ rises steadily during transition, the capital share (combining both the s-capital and c-capital shares), which (8) implies is equal to $1 - \psi(1 - \bar{i})$, steadily rises. In turn, (10) and (24) imply that the return to the fixed stock of c-capital, $r^C(t)$, is equal to:

$$(36) \qquad r^C(t) = \hat{\hat{K}}^S(t)^{\psi} \cdot a^S(0,t)^{\frac{\psi}{1-\psi}} \cdot \frac{a^C(0)^{1-\psi} \cdot (1-\psi)}{(K^C)^{\psi}}$$

implying that $r^C(t)$ rises in transition and steady-state when $\bar{i}(t) = 1$.

3. Extension

3.1. The Creation of New Tasks?

In the model in this paper, the task-continuum has a fixed upper-bound equal to one.[28] Yet in practice it may be that the task-continuum changes over time. As

[28] This is the particular assumption that leads Autor and Salomons (2017a,b) to observe that labour has "no place left to hide" in the model.

noted before, this possibility is captured in Acemoglu and Restrepo (2018a,b). A novel feature of their model is that the task-continuum is not fixed: new tasks are endogenously created in the direction of labour's comparative advantage. More specifically, as labour is displaced by capital, wages fall and the cost of producing output with labour also falls: this creates an incentive for firms to create entirely new tasks for the now-cheaper labour to do. And so, given this process, displaced workers are then able to perform those new tasks, offsetting the process of immiseration that I have identified in this paper.

However, as Acemoglu and Restrepo (2018a) also note, it is not inevitable that sufficient new tasks are created for labour to do. The authors explicitly identify a possible case where insufficient new tasks are created for labour – what they call the "horse equilibrium" (where human beings, like horses before them, are immiserated by technological progress) – but do not elaborate in detail on what the dynamics in this case would look like. The case arises in their framework when the long-run rental rate of capital is sufficiently low relative to the wage: in short, when there is little incentive for firms to create new tasks for labour to do, because capital is sufficiently cheap relative to labour. But given the assumptions made in Acemoglu and Restrepo (2018a), there are also further theoretical reasons to imagine that this case might occur, alongside those identified explicitly identified by the authors.

One restrictive assumption that is made in their model, for instance, is the claim that the production of new tasks for labour to do is linear in the allocation of resources to that process: but it seems intuitive to imagine that it may get *harder* over time to create new tasks for labour to do, and that production function should be characterised by diminishing marginal returns.[29] Another restrictive assumption is the claim that any new tasks are *necessarily* created in the direction of *labour's* comparative advantage: but, again, it seems intuitive to imagine that capital could become so productive relative to labour over time that firms instead instead face an incentive to create new tasks for *capital* to do. Both these possibilities – that it might get harder to find new things for labour to do over time, and that firms might instead face an incentive to create new tasks for capital to do – would make it less likely that sufficient tasks are create for displaced workers to do, and more likely that labour is immiserated by technological progress in their framework and the 'horse equilibrium' is achieved.

[29]This is equation (22) – the creation of new tasks, $\dot{N}(t)$, is linear in the allocation of scientists to the creation of new tasks, $S_N(t)$, with coefficient κ_N. It seems intuitive to think that a more reasonable production function might be $\dot{N}(t) = \kappa_N S_N(t)^\alpha$, where $0 < \alpha < 1$.

In turn, alongside these theoretical arguments for believing that insufficient tasks might be created for displaced workers to do, is empirical evidence that suggests, in practice, the process of new task creation may be slowing down. Acemoglu and Restrepo (2019a), for instance, argue that over the last three decades there has been a "striking" slowdown in the growth of labour demand: the per capita wage bill in the US grew only 1.33 percent per year on average from 1987-2017, though stagnated altogether since 2000. Their conclusion is that this deceleration in labour demand growth is, in part, due to "rapid automation that is not being counterbalanced by the creation of new tasks".

Together, then, these theoretical arguments and empirical observations show why it is important to understand in greater detail what happens in a setting where insufficient new tasks are created for labour to do, and to explore what other countervailing forces might offset the immiseration of labour in such a scenario. And the new model in this paper, with a fixed task-continuum, provides a tractable setting to explore that case. In turn, the new distinction between s-capital and c-capital shows that the behaviour of c-capital is critical in this setting: if no new tasks are created, then c-capital might provide an alternative countervailing force to offset the immiseration of labour.

3.2. The Role of C-Capital

In the model set out before, technological progress only takes place in s-capital, and only s-capital is accumulated. A reasonable extension is consider the case where there is technological progress also takes place in c-capital, and both types of capital are accumulated. However, in this more general case, it is no longer possible to identify a dynamic equilibrium with a steady state in which the rate of return on capital is constant. The reason is as follows. In a setting where both types of capital are accumulated, the following arbitrage condition must hold:

$$r^S(t) = r^C(t) = r(t)\ \forall t \tag{37}$$

implying that the return on each type of capital is the same. The final relative factor price derived in (10) implies that if the return on each type of capital is the same, as in (37), then the following condition must hold:

$$g^{K^S}(t) - g^{K^C}(t) = g^{\bar{i}}(t) \tag{38}$$

which implies that the difference in the growth rate in the accumulation of each type of capital, $g^{K^S}(t)$ and $g^{K^C}(t)$, must be equal to the growth rate in the cut-off, $g^{\bar{i}}(t)$. But if the return on s-capital, $r^S(t)$, is to remain constant in any steady-state, then the expression for $r^S(t)$ in (24) implies that the following condition must also hold:

$$(39) \qquad g^{K^S}(t) - g^{K^C}(t) = g^{\bar{i}}(t) + \frac{\psi}{1-\psi} \cdot g^S + g^C$$

where g^S and g^C are the growth rates in the productivity of s-capital and c-capital respectively. (39) and (38) are clearly incompatible – they require different differences between the growth rate in s-capital and c-capital. And so, is not possible to find a dynamic equilibrium where the arbitrage condition holds and the rate of return on capital is constant at the same time. Given the arbitrage condition must hold, this implies that, over time, the rate of return on capital must rise continually in a dynamic version of the new model where there is technological progress in both types of capital, and both types are also accumulated. In effect, this version of the model converges to an 'AK' model, where the K is composed of both K^S and K^C (a similar result is found in Peretto and Saeter 2013).

However, despite the absence of a dynamic equilibrium in the case where c-capital is also endogenously accumulated, is still possible to perform comparative statics with c-capital at any moment during transition in the original dynamic equilibrium, where only s-capital is endogenously accumulated. To do this, note that an explicit expression for $w(t)$ can be identified from (10) and (36):

$$(40) \qquad w(t) = (a^C(0,t) \cdot K^C(t))^{1-\psi} \cdot \frac{\hat{\hat{K}}^S(t)^{\psi} \cdot a^S(0,t)^{\frac{\psi}{1-\psi}} \cdot \psi \cdot (1 - \bar{i}(t))}{L}$$

From (40), Proposition 3 follows.

PROPOSITION 3: *In any period t, the wage is an increasing function of both the level and productivity of c-capital:*

$$\frac{\partial w(t)}{\partial a^C(0,t)} > 0, \quad \frac{\partial w(t)}{\partial K^C(t)} > 0$$

This result is new and matters because it suggests that, in the absence of a process that creates new tasks for displaced workers to do, c-capital can act as an alternative countervailing force to the immiseration of labour, either by becoming more productive or more abundant.

3.3. The 'Right' Kind of AI

In Acemoglu and Restrepo (2019b), the authors distinguish between two different kinds of artificial intelligence (AI): the "right" kind of AI, which are systems and machines that create new tasks for workers to do; and the "wrong" type, which are those that displace workers without creating new tasks for them to do. The new model in this paper builds on this distinction, and shows that in a setting where the process of task creation is ineffective or absent, then the "right" v. "wrong" distinction takes on an alternative meaning – the "right" kind of AI are systems and machines that with c-capital properties, and the "wrong" kind of AI are those with s-capital properties.

From a policy standpoint, Acemoglu and Restrepo (2018a, b, 2019b) demonstrate the importance of understanding the dynamics of new task creation – and what sort of interventions might support that process. The lesson from this paper, though, is that is also very important to understand the nature of the technologies themselves – and what interventions might encourage the use of those with c-capital rather than s-capital properties. And this insight is particularly relevant for two reasons. First, as before, there are theoretical arguments and empirical observations that show it is not at all inevitable that sufficient new tasks will be created for labour to do – and so it is important to explore other countervailing forces against the immiseration of the labour. But secondly, there is also a growing body of evidence suggesting that recent technological change may indeed have shifted towards s-capital and away from c-capital. For instance, Autor and Salomons (2018) conclude that "automation... [has] become in recent decades less labour-augmenting and more labour-displacing" – this alternative countervailing force also appears to be weakening.

3.4. Conclusion

Autor (2014) captured the traditional case for optimism in the early task-based literature with the observation that "tasks that cannot be substituted by computerization are generally complemented by it"; and Autor (2015) restated it, explaining how "tasks that cannot be substituted by automation are generally complemented by it". The analysis in this paper fit with these early claims: those tasks that cannot be automated are complemented by c-capital, and the value of those tasks increases as the quantity or productivity of c-capital increases. However, the new model does challenge the assumption that labour will indefinitely be best placed to perform those complemented tasks. This is the new role for s-capital; any increase in its quantity or productivity erodes the set of types of q-complemented tasks in which labour retains the comparative advantage, and real wages fall. This is the process of task encroachment, where labour is forced to specialise in a diminishing set of tasks. In Acemoglu and Restrepo (2018a, b), the creation of new tasks acts as a countervailing force against this immiseration of labour. In this paper, by using a new distinction between two different types of capital, I identify an alternative countervailing force – increases in the quantity or productivity of c-capital.

This paper is an additional step towards a more general understanding of the full set of countervailing forces that might offset the immiseration of labour if systems and machines continue to become more capable over time. And the framework set out provides various routes for further exploration. First, the process of task encroachment explored is very simple, where s-capital gradually takes on more 'complex' tasks, and the only possible variation is in the speed of this process (determined by the magnitude of g in 12). However, in practice it is unlikely that encroachment takes place in such a deterministic way, and exploring the different forms that this process of task encroachment could take is an interesting task. Secondly, the model relies on the assumption that there is no growth in the productivity of labour. But growth in the productivity of labour might also act as a countervailing force against the immiseration of labour, by both allowing labour to retain the comparative advantage in a broader set of tasks, and by raising the demand fo labour to perform those tasks.[30]

A further step is the exploration of other forms for the factor-based production function for tasks in (3) and the task-based production function for goods in (2). For instance, the model relies on the assumption that (2) is Cobb-Douglas, so expenditure shares on the two different types of tasks, $z_1(i)$ and $z_2(i)$, remain fixed – but it is possible in practice that the production function takes a different form. Take, for instance, a production function where expenditure shares on the two different types of tasks *do* change in response to changes in the relative price of those tasks. And suppose relative price changes cause expenditure shares to increase on tasks performed by labour. This would help to offset the process of

[30]The static model in Section 1 shows that an increase in productivity of labour across the task-continuum causes a fall in $\bar{i}$, and an increase in $w(t)$ in (37) – both by increasing $\hat{\hat{K}}^S(t)$ (recall that $\bar{i}$ enters the denominator of the expression for 'effective' c-capital in (21)) and by increasing the $1-\bar{i}(t)$ term.

immiseration – labour may be confined to a shrinking range of types of tasks, but there would also be a greater relative demand for those residual tasks. In the opposite case, where relative price changes caused expenditure shares to decrease on tasks performed by labour, this would compound the process of immiseration. These observations suggest that further work is needed to understand the different ways in which the latest technologies combine with labour in production.

A final step concerns the demand-side. An emerging set of arguments for remaining optimistic about the threat of automation appeal to consumer tastes and preferences. The first is a claim that demand for goods will *rise* and displaced workers can be employed to meet that *increase in demand for goods*. For instance, Autor (2015) notes "I think that people are extremely unduly pessimistic ... as people get wealthier, they tend to consume more, so that also creates demand", Summers (2013), that "[t]he stupid people thought that automation was going to make all the jobs go away ... the smart people understood that when more was produced, there would be more income and therefore there would be more demand", and Kenneth Arrow, that "the economy does find other jobs for workers. When wealth is created, people spend their money on something" (Harden 1982). The second form of the argument is a claim that demand for goods will *change* and displaced works can be employed to meet that *change in demand across goods*. For instance, Mokyr et al. (2015) "[t]he future will surely bring new products that are currently barely imagined, but will be viewed as necessities by the citizens of 2050 or 2080" and Autor and Dorn (2013b), that the economy will "generate new products and services that raise national income and increase overall demand for labour in the economy".

These arguments are appealing and intuitive. But at present, the most widely used models that explore the effect of technological change on the labour market cannot capture them. The problem is that these models tend to only have a unique final good and so no role for tastes and preferences (see, for instance: Autor, Levy and Murnane 2003; Acemolgu and Autor 2011; and Acemoglu and Restrepo 2018a, b).[31] However, the new model in this paper does provide a clear way to explore these arguments in a formal way. Because preferences are Cobb-Douglas in (1), expenditure shares across goods do not change, either due to income or price effects. Yet with more complex preferences, it is possible to show that expenditure shares can change over time in this model. In turn, it is also

[31] The notable exception to this is Autor and Dorn (2013a), which does have a role for the demand-side. However, it is a limited one; the model only has two goods. Bessen (2018) makes a similar observation to this one.

possible to show that if consumer expenditure shifts towards goods that require types of tasks in which labour retains a comparative advantage, then this can also act to countervail immiseraiton as well (see, for instance, Susskind 2018).

Each of these steps, then, is a possible direction for further research.

4. Appendix (For Online Publication)

4.1. Deriving the Hamiltonian

To find this steady-state, I used a Hamiltonian. But this is an unconventional Hamiltonian – there is one state variable, $K^S(t)$, one co-state variable, $\mu(t)$, and a range of choice variables, $x(i,t)$. To show that the traditional Hamiltonian approach applies in this setting, I can derive the Hamiltonian and accompanying first order conditions explicitly. The aim is to maximise:

(41)
$$\int_0^\infty e^{-\rho t}\left[\int_0^1 \theta(i)\ln x(i,t)\,di + \mu(t)\left[Y(t) - \int_0^1 x(i,t)\cdot p(i,t)\,di - \dot{K}^S(t)\right]\right] dt$$

where $Y(t)$ is defined as before. Integrating by parts implies:

(42)
$$\begin{aligned}\int_0^\infty e^{-\rho t}\cdot\mu(t)\cdot\dot{K}^S(t)dt &= \left[e^{-\rho t}\cdot\mu(t)\cdot K^S(t)\right]_{t=0}^{t=T} - \int_0^\infty K^S(t)\frac{\partial\left[\mu(t)\cdot e^{-\rho t}\right]}{\partial t}\,dt \\ &= e^{-\rho T}\cdot\mu(T)\cdot K^S(T) - \mu(0)\cdot K^S(0) - \int_0^\infty K^S(t)\cdot e^{-\rho t}\left[\dot{\mu}(t) - \rho\cdot\mu(t)\right] dt\end{aligned}$$

and since:

$$A = e^{-\rho T}\cdot\mu(T)\cdot K^S(T) - \mu(0)\cdot K^S(0) \tag{43}$$

where A is a constant, it follows that (41) can be re-written:

(44)
$$\int_0^\infty e^{-\rho t}\left[\int_0^1 \theta(i)\ln x(i,t)\,di + \mu(t)\left[Y(t) - \int_0^1 x(i,t)\cdot p(i,t)\,di\right] + K^S(t)\cdot\left[\dot{\mu}(t) - \rho\cdot\mu(t)\right]\right] dt \\ -A$$

It follows that to maximise (44) over time the following expression must be maximised for each given t:

$$\int_0^1 \theta(i) \ln x(i,t)\, di + \mu(t) \left[Y(t) - \int_0^1 x(i,t) \cdot p(i,t)\, di \right] + K^S(t) \cdot [\dot{\mu}(t) - \rho \cdot \mu(t)] \tag{45}$$

or more concisely the following must be maximised in each period t:

$$H + K^S(t) \cdot [\dot{\mu}(t) - \rho \cdot \mu(t)] \tag{46}$$

where H is the current value Hamiltonian. And so, in the traditional way, to maximise the expressions in (45) and (46) a traditional set of first order conditions follow:

$$\begin{aligned} \frac{\partial H}{\partial x(i,t)} &= 0 \quad \forall i \\ \frac{\partial H}{\partial K^S(t)} + \dot{\mu}(t) - \rho \cdot \mu(t) &= 0 \end{aligned} \tag{47}$$

These are the first order conditions that I use in this paper. This derivation is based on Wren-Lewis (2012).

4.2. Assumption 3

To derive the condition stated in Assumption 3, take two arbitrary levels of i such that $\tilde{i} < \tilde{\tilde{i}}$. Suppose Assumption 2 initially holds so that at $t = 0$:

$$A(\tilde{i}, 0) \leq A(\tilde{\tilde{i}}, 0) \tag{48}$$

Given the definition of $A(i,t)$ in (4) this implies:

$$a^S(\tilde{\tilde{i}}, 0) \leq a^S(\tilde{i}, 0) \cdot \frac{a^L(\tilde{\tilde{i}}, 0)}{a^L(\tilde{i}, 0)} \tag{49}$$

For Assumption 2 to continue hold over time, which is the condition stated in Assumption 3, it must be that $\forall t$:

$$a^S(\tilde{\tilde{i}}, t) \leq a^S(\tilde{i}, t) \cdot \frac{a^L(\tilde{\tilde{i}}, t)}{a^L(\tilde{i}, t)} \tag{50}$$

To see that the growth process in (12) maintains this condition, first note that (12) implies:

$$a^S(i,t) = a^S(i,0) \cdot e^{gt} \tag{51}$$

Also note that (49) and (51) imply:

$$\begin{aligned} a^S(\tilde{\tilde{i}},0) \cdot e^{gt} &\leq a^S(\tilde{i},0) \cdot e^{gt} \cdot \frac{a^L(\tilde{\tilde{i}},0)}{a^L(\tilde{i},0)} \\ a^S(\tilde{\tilde{i}},t) &\leq a^S(\tilde{i},t) \cdot \frac{a^L(\tilde{\tilde{i}},0)}{a^L(\tilde{i},0)} \end{aligned} \tag{52}$$

and so it follows that given (52) holds, then Assumption 3, captured by the condition in (50), also holds, since the productivities of labour do not change over time.

4.3. Law of Motion for $x(i,t)$

From (15) it follows that:

$$x(i,t) = \frac{1}{\mu(t) \cdot p(i,t)} \tag{53}$$

And from (16) that:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -r^S(t) + \rho \tag{54}$$

It follows from (53) that:

$$\frac{\dot{x}(i,t)}{x(i,t)} = -\frac{\dot{p}(i,t)}{p(i,t)} - \frac{\dot{\mu}(t)}{\mu(t)} \tag{55}$$

(54) and (55) therefore imply that:

$$\frac{\dot{x}(i,t)}{x(i,t)} = -\frac{\dot{p}(i,t)}{p(i,t)} + r^S(t) - \rho \tag{56}$$

4.4. Derivation of $r^S(t)$ and $\hat{Y}(t)$ in terms of $\hat{\hat{K}}^S(t)$

Expression for $p(i,t)$: The first step in deriving $r^S(t)$ and $\hat{Y}(t)$ in terms of $\hat{\hat{K}}^S(t)$ is to find an expression for $p(i,t)$. Suppose that $i(t) \in [0, \bar{i}(t)]$. This implies that $x(i,t)$ is produced by s-capital and c-capital:

$$x(i,t) = \left[a^S(i,t)K^S(i,t)\right]^{\psi} \left[a^C(i,t)K^C(i,t)\right]^{1-\psi} \quad \forall i(t) \in [0, \tilde{i}(t)] \tag{57}$$

This implies that the marginal product of s-capital in producing $x(0,t)$ is:

$$\begin{aligned} MPK^S(i,t) &= a^S(i,t) \cdot \psi \left[a^S(i,t)K^S(i,t)\right]^{\psi-1} \left[a^C(i,t)K^C(i,t)\right]^{1-\psi} \\ &= \psi \cdot \frac{x(i,t)}{K^S(i,t)} \end{aligned} \tag{58}$$

and the marginal product of c-capital in producing $x(i,t)$ is:

$$\begin{aligned} MPK^C(i,t) &= a^C(i,t) \cdot (1-\psi) \cdot \left[a^L(i,t)L^L(i,t)\right]^{\psi} \left[a^C(i,t)K^C(i,t)\right]^{-\psi} \\ &= (1-\psi) \cdot \frac{x(i,t)}{K^C(i,t)} \end{aligned} \tag{59}$$

Given perfectly competitive profit-maximising firms, the price of each of these factors – $r^S(t)$ and $r^C(t)$ – must be equal to their respective marginal revenue products:

$$\begin{aligned} r^S(t) &= p(i,t) \cdot \psi \cdot \frac{x(i,t)}{K^S(i,t)} \qquad \forall i(t) \in [0, \bar{i}(t)] \\ r^C(t) &= p(i,t) \cdot (1-\psi) \cdot \frac{x(i,t)}{K^C(i,t)} \qquad \forall i(t) \in [0, 1] \end{aligned} \tag{60}$$

These can be re-arranged:

$$\begin{aligned} K^S(i,t) &= p(i,t) \cdot \psi \cdot \frac{x(i,t)}{r^S(t)} \qquad \forall i(t) \in [0, \bar{i}(t)] \\ K^C(i,t) &= p(i,t) \cdot (1-\psi) \cdot \frac{x(i,t)}{r^C(t)} \qquad \forall i(t) \in [0, 1] \end{aligned} \tag{61}$$

and substituting these expressions for $K^S(i,t)$ and $K^C(i,t)$ into (57) implies:
(62)

$$\begin{aligned} x(i,t) &= \left[a^S(i,t) \cdot p(i,t) \cdot \psi \cdot \frac{x(i,t)}{r^S}\right]^{\psi} \left[a^C(i,t) \cdot p(i,t) \cdot (1-\psi) \cdot \frac{x(i,t)}{r^C(t)}\right]^{1-\psi} \\ p(i) &= \left[\frac{r^S}{\psi \cdot a^S(i)}\right]^{\psi} \left[\frac{r^C}{(1-\psi) \cdot a^C(i)}\right]^{1-\psi} \end{aligned}$$

(62) is therefore the $p(i,t)$ for any good $i(t) \in [0, \bar{i}(t)]$. A similar exercise to derive $L(i,t)$ provides $p(i,t)$ for those remaining goods $i(t) \in [\bar{i}(t), 1]$ produced by labour with c-capital.

Expression for $r^S(t)$ in terms of $\hat{\hat{K}}^S(t)$: To now find $r^S(t)$ in terms of $\hat{\hat{K}}^S(t)$, first note that (62) implies that $p(0,t)$, the price of the numeraire good, is equal to:

$$p(0,t) = \left[\frac{r^S(t)}{\psi \cdot a^S(0,t)}\right]^{\psi} \left[\frac{r^C(t)}{(1-\psi) \cdot a^C(0,t)}\right]^{1-\psi} \tag{63}$$

Substituting the expression for $r^C(t)$ in terms of $r^S(t)$ that follows from (10), and using the price normalisation that $p(0,t) = 1$, (63) implies that $r^S(t)$ can be re-written as:

$$\begin{aligned} r^S(t) &= \left[a^S(0,t)\right]^{\psi} \left[a^C(0)\right]^{1-\psi} \cdot [\bar{i}(t)]^{1-\psi} \cdot \left[\frac{1}{K^S(t)}\right]^{1-\psi} \cdot K^{C(1-\psi)} \cdot \psi \\ &= \hat{\hat{K}}^S(t)^{\psi-1} \cdot D \end{aligned} \tag{64}$$

where D is a positive constant equal to $\left(a^C(0) \cdot K^C\right)^{1-\psi} \cdot \psi$.

Expression for $\hat{Y}(t)$ in terms of $\hat{\hat{K}}^S(t)$ – Now consider $\hat{Y}(t)$ in terms of $\hat{\hat{K}}^S(t)$. Note that the structure of production in (2) and (3) implies that total s-capital income is equal to:

$$r^S(t) \cdot K^S(t) = \bar{i}(t) \cdot \psi \cdot Y(t) \tag{65}$$

and so substituting in $r^S(t)$ from (64) it follows that:

$$\hat{Y}(t) = \hat{\hat{K}}^S(t)^{\psi-1} \cdot D \cdot \hat{K}^S(t) \cdot \frac{1}{\bar{i}(t)} \cdot \frac{1}{\psi}$$

(66)

$$= \hat{\hat{K}}^S(t)^{\psi} \cdot \frac{D}{\psi}$$

4.5. Law of Motion for $\hat{\hat{K}}^S(i,t)$

Using (25) and (26), (23) can be written as:

$$\frac{\dot{\hat{\hat{K}}}^S(t)}{\hat{\hat{K}}^S(t)} = \frac{\hat{\hat{K}}^S(t)^{\psi} \cdot \frac{D}{\psi} - \hat{c}(t)}{\hat{K}^S(t)} - \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^S(t)} \cdot \dot{\hat{\hat{K}}}^S(t) \cdot \frac{1}{\bar{i}(t)} + \frac{\psi}{1-\psi} \cdot g\right] \tag{67}$$

And so:

(68)

$$\dot{\hat{\hat{K}}}^S(t) = \frac{1}{\bar{i}(t)} \cdot \left[\hat{\hat{K}}^S(t)^{\psi} \cdot \frac{D}{\psi} - \hat{c}(t)\right] - \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^S(t)} \cdot \dot{\hat{\hat{K}}}^S(t) \cdot \frac{1}{\bar{i}(t)} + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{K}}^S(t)$$

$$= \frac{\hat{\hat{K}}^S(t)^{\psi} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g \cdot \hat{\hat{K}}^S(t) - \hat{c}(t)}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^S(t)} \cdot \hat{\hat{K}}^S(t)}$$

It follows that:

$$\frac{\dot{\hat{\hat{K}}}^S(t)}{\hat{\hat{K}}^S(t)} = \frac{\hat{\hat{K}}^S(t)^{\psi-1} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\hat{c}(t)}{\hat{\hat{K}}^S(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^S(t)} \cdot \hat{\hat{K}}^S(t)} \tag{69}$$

4.6. Transition Path

To show that $\bar{i}(t)$ increases as $\hat{\hat{K}}^S(t)$ increases on the transition path, I first derive the following relationship between the growth rates in $\bar{i}(t)$, $K^S(t)$ and g:

$$g^{\bar{i}}(t) = \phi(t)\left[g^{K^S}(t) + g\right] \tag{70}$$

where:

$$(71) \qquad \phi(t) = \frac{1-\bar{i}(t)}{[\bar{i}(t)-\bar{i}(t)^2]\cdot y(t)+1} \quad \text{and} \quad y(t) = \left[\frac{a_1^H(\bar{i}(t),t)}{a^L(\bar{i}(t),t)} - \frac{a_1^S(\bar{i}(t),t)}{a^S(\bar{i}(t),t)}\right]$$

and $\phi(t)$ has the three properties: $\phi(t) \geq 0$; $\phi(t) \leq 1$; and $\lim_{\bar{i}(t)\to 1}\phi(t) = 0$. The expression in (70) implies that the growth rate in the equilibrium cut-off $\bar{i}(t)$, $g^{\bar{i}}(t)$, is proportional to the sum of the growth rate in the s-capital stock $g^{K^S}(t)$ and the productivity of $K^S(t)$, $\frac{\psi}{1-\psi}\cdot g$ where the constant of proportionality $\phi(t)$ is time dependent with the three features set out above. I now derive this expression for $g^{\bar{i}}(t)$ and these three properties of $\phi(t)$.

Deriving $g^{\bar{i}}(t)$: To derive the expression for $g^{\bar{i}}(t)$, note that the equilibrium condition, $A(\bar{i}(t),t) = B(\bar{i}(t),t)$ can be re-arranged as:

$$(72) \qquad a^L(\bar{i}(t),t)\cdot\bar{i}(t) = [1-\bar{i}(t)]\cdot a^S(\bar{i}(t),t)\cdot\frac{K^S(t)}{L}$$

Taking time derivatives of (72) implies a further condition that must hold $\forall t$:

(73)

$$\dot{\bar{i}}(t)\cdot\frac{a_1^H(\bar{i}(t),t)}{a^L(\bar{i}(t),t)} + \frac{a_2^H(\bar{i}(t),t)}{a^L(\bar{i}(t),t)} + \frac{\dot{\bar{i}}(t)}{\bar{i}(t)} = -\frac{\dot{\bar{i}}(t)}{[1-\bar{i}(t)]} + \dot{\bar{i}}(t)\cdot\frac{a_1^S(\bar{i}(t),t)}{a^S(\bar{i}(t),t)} + \frac{a_2^S(\bar{i}(t),t)}{a^S(\bar{i}(t),t)} + \frac{\dot{K}^S(t)}{K^S(t)}$$

and substituting in the relevant expressions for the growth rates – $g^{K^S}(t)$ the growth rate in the stock of s-capital, $g^{\bar{i}}(t)$ the growth rate in the equilibrium cut-off $\bar{i}(t)$, and g the growth rate in the productivity of s-capital – implies that:

$$(74) \qquad g^{\bar{i}}(t)\left[\bar{i}(t)\cdot\left[\frac{a_1^H(\bar{i}(t),t)}{a^L(\bar{i}(t),t)} - \frac{a_1^S(\bar{i}(t),t)}{a^S(\bar{i}(t),t)}\right] + \frac{1}{[1-\bar{i}(t)]}\right] = g + g^{K^S}(t)$$

and so:

$$(75) \qquad g^{\bar{i}}(t) = \phi(t)\left[g^{K^S}(t) + g\right]$$

where:

(76)

$$\phi(t) = \left[\bar{i}(t)\cdot\left[\frac{a_1^H(\bar{i}(t),t)}{a^L(\bar{i}(t),t)} - \frac{a_1^S(\bar{i}(t),t)}{a^S(\bar{i}(t),t)}\right] + \frac{1}{[1-\bar{i}(t)]}\right]^{-1} = \frac{1-\bar{i}(t)}{[\bar{i}(t)-\bar{i}(t)^2]\cdot y(t)+1}$$

Deriving the Three Properties of $\phi(t)$: First to see that $\phi(t) \geq 0$ note that the weakly positive slope of $A(i,t)$ implies $\forall i, t$:[32]

$$A_1(i,t) \geq 0 \tag{77}$$

which, given the definition of $A(i,t)$, can be re-written as:

$$\frac{a_1^H(i,t) \cdot a^S(i,t) - a_1^S(i,t) \cdot a^L(i,t)}{[a^S(i,t)]^2} \geq 0 \tag{78}$$

Since the denominator of the expression in (78) is always positive, this implies that the following must hold:

$$\begin{gathered} a_1^H(i,t) \cdot a^S(i,t) \geq a_1^S(i,t) \cdot a^L(i,t) \\ \frac{a_1^H(i,t)}{a^L(i,t)} \geq \frac{a_1^S(i,t)}{a^S(i,t)} \end{gathered} \tag{79}$$

Therefore so long as the $A(i,t)$ schedule is weakly positively sloped, $y(t)$ remains weakly positive, the denominator in the expression for $\phi(t)$ in (76) remains positive, and so $\phi(t)$ also remains weakly positive. To see the second property, that $\phi(t) \leq 1$, consider the following proof by contradiction. Assume instead that $\phi(t) > 1$. The expression for $\phi(t)$ in (76) then implies:

$$\begin{aligned} \bar{i}(t) \cdot \left[\frac{a_1^H(\bar{i}(t),t)}{a^L(\bar{i}(t),t)} - \frac{a_1^S(\bar{i}(t),t)}{a^S(\bar{i}(t),t)}\right] + \frac{1}{[1-\bar{i}(t)]} &< 1 \\ \bar{i}(t) \cdot [1-\bar{i}(t)] \cdot y(t) &< 1 - \bar{i}(t) - 1 \\ \bar{i}(t) \cdot y(t) - \bar{i}(t)^2 \cdot y(t) &< -\bar{i}(t) \end{aligned} \tag{80}$$

But since $\bar{i}(t) \in [0,1]$ this condition cannot hold. If $\bar{i}(t) = 0$, then the result is a contradiction since (80) requires that $0 < 0$. Similarly if $\bar{i}(t) > 0$ then (80) requires:

$$y(t) < -\frac{1}{1-\bar{i}(t)} < 0 \tag{81}$$

which is not possible since $\bar{i}(t) \in [0,1]$ and $y(t) \geq 0$. The third property, that $\lim_{\bar{i}(t)\to 1} \phi(t) = 0$, follows from the fact that:

[32] This is Assumptions 2 and 3.

$$\lim_{\bar{i}(t)\to 1}\left[\frac{1}{1-\bar{i}(t)}\right]=\infty \tag{82}$$

and that:

$$\lim_{\bar{i}(t)\to 1}\left[\bar{i}(t)\cdot\left[\frac{a_1^H(\bar{i}(t),t)}{a^L(\bar{i}(t),t)}-\frac{a_1^S(\bar{i}(t),t)}{a^S(\bar{i}(t),t)}\right]\right]\geq 0 \tag{83}$$

And so $\lim_{\bar{i}(t)\to 1}\phi(t)=0$, given (76), (82) and (83).

$\bar{i}$ *During Transition*: The proof that $\frac{d\bar{i}(t)}{d\hat{\hat{K}}^S(t)}\geq 0$ during the conjectured transition in Figure 5 now follows from two relationships. The first is implied by the fact that if $\hat{\hat{K}}^S(t)$ is increasing, the definition in (21) then implies:

$$g^{K^S}(t)>g^{\bar{i}}(t)+\frac{\psi}{1-\psi}\cdot g \tag{84}$$

where $g^{K^S}(t)$ is again the growth rate in the stock of s-capital and $g^{\bar{i}}(t)$ the growth rate in the equilibrium cut-off $\bar{i}(t)$. The second relationship is that set out in (70). Combining these two sets of relationships implies that if $\hat{\hat{K}}^S(t)$ is increasing then:

$$\frac{1-\phi(t)}{\phi(t)}\cdot g^{\bar{i}}(t)>\frac{1}{1-\psi}\cdot g \tag{85}$$

And so:

$$g^{\bar{i}}(t)>\frac{\phi(t)}{(1-\psi)\cdot(1-\phi(t))}\cdot g \tag{86}$$

Given the properties of $\phi(t)$ derived before, and that $g>0$, this implies that if $\hat{\hat{K}}^S(t)$ is increasing then:

$$g^{\bar{i}}(t)>0 \tag{87}$$

i.e. $\bar{i}(t)$ is also increasing.

5. References

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